Embeddability of infinite graphs

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Embeddability in the plane: Kuratowski, Wagner

Theorem (Kuratowski, 1930)

A **finite** graph G is embeddable in the plane if and only if it does not contain a subgraph homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$.

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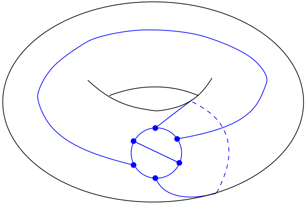
Every compact surface has a "Wagner's Theorem":

Theorem (Robertson and Seymour, 1990)

For every compact surface there is a finite list of graphs such that a graph G is embeddable in this surface if and only if it does not contain any of these as a minor.

$g+1$ disjoint Kuratowski graphs: natural obstacle for embedding in genus g

$K_{3,3}$ and K_5 are the **Kuratowski graphs**.



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The compact surface of genus g can host the disjoint union of **g** Kuratowski graphs (and no more)

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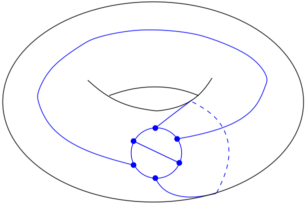

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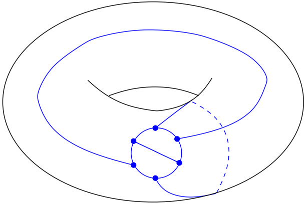

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Is this **the** reason why the genus of a graph grows?

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Robertson and Seymour (unpublished)

There is a function $f(g)$ tending to infinity so that, if a graph G does not embed in any surface of Euler characteristic at least $2-2g$, then G has one of the following graphs as a minor:

1. $f(g)$ disjoint copies of either $K_{3,3}$ or K_5;
2. $f(g)$ copies of either $K_{3,3}$ or K_5 that are disjoint except for a common vertex;
3. $f(g)$ copies of either $K_{3,3}$ or K_5 that are disjoint except for two common vertices; or
4. $K_{3,f(g)}$

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If we restrict ourselves to orientable surfaces, then we have to add the $f(g)$-projective grid to the list.

Our problem

Which graphs do not embed into **any** surface of bounded genus?

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Plausible answer, in view of the Robertson-Seymour result

Those that contain as a minor either:

- Infinitely many "sufficiently disjoint" $K_{3,3}$'s or K_5's.
- $K_{3,\infty}$ (yes, abuse of notation)

Some previous fine-tunning

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Thus, it all boils down to:

Question

Which countable graphs embed in **some** surface of bounded genus?

Our result: embeddability in bounded genus

Theorem (Christian, Richter, and S., 2011^{+})

A countable graph G embeds in some (orientable) surface of bounded genus if and only if G does not contain as a minor any of:

1. infinitely many disjoint copies of either $K_{3,3}$ or K_5;
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A (slightly surprising?) consequence

There is no distinction between embeddability in some orientable surface and embeddability in some surface. In other words, no graph can embed in some (non-orientable) surface and have arbitrarily large projective grids.

The interesting direction

The "only if" part is easy: a graph with infinitely many (sufficiently disjoint) copies of $K_{3,3}$ or K_5, or with $K_{3,\aleph_0}$, cannot be embedded in any surface of bounded genus.

The "if" part is the interesting one.

Getting started

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A graph is **good** if it can be embedded in some surface.

Otherwise it is **bad**.

A bad G has infinitely many disjoint copies of $K_{3,3}$ or K_5 or:

There is a $J \subseteq G$, and a vertex u_1 of J, such that J is bad and $J - u_1$ is good.

1. Let $G_0 := G$, and as long as G_i has a subgraph H_{i+1} (may choose finite, if one exists) that contracts to $K_{3,3}$ or K_5, set $G_{i+1} := G_i - V(H_{i+1})$.

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2. If for every positive i, H_i exists, then we are done (G contains as a minor infinitely many disjoint copies of either $K_{3,3}$ or K_5). Thus we may assume that for some i, G_i has no Kuratowski minor; so G_i is planar. Note that G_i is obtained from G by the deletion of finitely many vertices $v_1, v_2, \ldots, v_k$.

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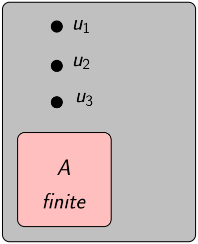
3. For $j = 1, 2, \ldots, k$, consider $G^j := G - \{v_{j+1}, v_{j+2}, \ldots, v_k\}$. There is a least j so that G^j does not embed in any surface. Set $J_0 := G^j$, and $u_1 := v_j$. Thus J_0 is a subgraph of G that does not embed in any surface (J_0 is bad), yet $J_0 - u_1$ does ($J_0 - u_1$ is good).

REPEAT AND GET: Either G has one of the listed minors or $\exists$:



$M \subseteq G$

- M is bad
- $M - u_i$ is good for each i
- $M - A$ has no subdivision of $K_{1,3}$ with u_1, u_2, u_3 as the degree 1 vertices.

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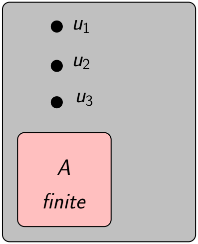

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IF THIS HAPPENS: Every component of $M - (A \cup \{u_1, u_2, u_3\})$ attaches to at most two of u_1, u_2, u_3. Let $N_{j,k}$ be the subgraph of M induced by the vertices in $A \cup \{u_j, u_k\}$ and all components of $M - (A \cup \{u_1, u_2, u_3\})$ that attach to at most u_j and u_k.

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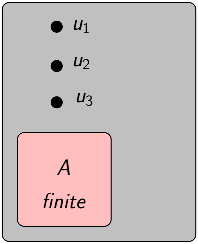

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Each of $N_{1,2}$, $N_{1,3}$, and $N_{2,3}$ embeds in some surface. . . combine these embeddings to obtain an embedding of M (contradiction!).

Last slide

Continuing in the countably infinite theme...

> *A countably infinite number of men went into a pub. The first one ordered a pint. The second ordered a half-pint. The third ordered a quarter of a pint ... The barkeeper, with a face full of disgust, finally poured two pints and put them on the bar and said, "It's good when people know their limits."*

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Thanks for your attention!